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Scale-dependent functions in statistical hydrodynamics: a functional analysis point of view

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Abstract. Most of dynamic systems which exhibit chaotic behavior are also known to posses self-similarity and manifest strong fluctuations of all possible scales. The meaning of this terms is not always same. In present note we make an attempt to formulate the problem in the framework of functional analysis. The statistical hydrodynamics is taken as a vivid physical example. The links to wavelet analysis are presented.

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1 Introduction

Self-similarity and scale-dependence come along in many physical phenomena. One of the most known examples is the hydrodynamics. The incompressible fluid flow described by the Navier-Stokes equation

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \nu\Delta\mathbf{v}, \quad \nabla\cdot\mathbf{v} = 0,$$

shows an intrinsic chaotic behavior. Formally, the solution $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ is a continuous differentiable function of its arguments. In fact its behaviour is so irregular, that may be strongly dependent on the grid size. That is why it is more reasonable to consider $\mathbf{v} = \mathbf{v}(\mathbf{x}, t; \Delta x, \Delta t)$, taken separately at different space and resolutions $(\Delta x, \Delta t)$. Physically, it is also more meaningful to consider a resolution-dependent function, since the vector field $\mathbf{v}(x, t)$ is measured as an average over a physical volume $(\Delta x)^D \Delta t$: this averaging is meaningless at the limit of infinite resolution $(\Delta x \to 0)$. Both the experimental observations in wind tunnels (the longtitudinal velocity measurements) and numerical simulations show that the turbulent velocity field resembles the Brownian motion

$$\langle |v(x + l) - v(x)|^2\rangle \sim l^{2/3},$$

exactly as it was inferred by Kolmogorov from simple dimensional consideration [6].

In practical calculations the scale dependence of the field $v(x)$ is usually evaluated either by the multifractal formalism (often with a help of wavelets) or by means of renormalization group (RG) technique, with the cutoff momentum understood as resolution parameter. So, at least up to the authors knowledge, the analysis of the scale-dependent properties of singular signals is usually performed over $v(x) \in L^2$, *where* $v(x)$ *is assumed to be measured at certain resolution, but this fact is nowhere clearly fixed mathematically.*

In the present paper I make an attempt to consider the problem from the functional analysis point of view and clarify how the object of the form $v(x, \Delta x)$ — the meaning of which is physically clear — can be treated from mathematical standpoint. The statistical hydrodynamics is referred to as a most vivid physical example, however the same technique can be also applied to the Brownian motion and signal processing.

2 Problem

Self-similarity is a synonym of scale-invariance. To be scale-invariant means to have same properties at different scales. Classical fractals are scale-invariant by construction. Self-similarity means to be similar (not necessary exactly the same) at different scales. Brownian motion is self-similar: if we look at the trajectory of Brownian particle at different resolutions of a microscope we will observe more or less the same picture. As physical systems are considered, the word self-similarity is more frequently attributed to their dynamics than geometry.

The self-similarity of hydrodynamic velocity field fluctuations $\langle(\delta v(l))^2\rangle \sim l^{2/3}$ is attributed to the behavior of turbulent velocity field measured at different spatial scales. For hydrodynamic velocity field it is physically clear, that the measurement at scale l_0 necessarily implies averaging of molecular velocities over certain space domain of typical size l_0. This procedure can be generalized

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to “an averaging of a function up to scale l” [2]

$$\phi_l(\mathbf{x}) = l^{-D}\int_{|y|<l}\phi(\mathbf{x}-\mathbf{y})d^D\mathbf{y}. \quad (1)$$

There are at least two conjectures here

1. The existence of “true” (no-scale) field $\phi_l(\mathbf{x}) : l \to 0$.
2. The homogeneity of the measure $d\mu(\mathbf{y}) = d^D\mathbf{y}$.

Physically, it is quite clear, that two different fields $\phi_l(\mathbf{x})$ and $\phi_{l'}(\mathbf{x})$ live in different functional spaces if $l \neq l'$. It is meaningless, say, to subtract their values. Therefore, the velocity field of hydrodynamic turbulence is something more than a random vector field defined on $R^D \times R$.

To characterize the turbulent velocity at certain point $\mathbf{x}$ we ought to know the collection of velocity values at a set of scales labeled by l $\{\phi_l(\mathbf{x})\}$. The set of scales may be countable

$$l = l_0, kl_0, k^2l_0, k^3l_0, \ldots,$$

say $k = 1/2$ for period-doubling decomposition, or continuous.

To characterize this set it was proposed in [2] to use a collection of unit fields at different scales – a “reference field” $\{\mathcal{R}_l(x)\}$. The principal question arising here, is how to describe the interaction of the fluctuations of different scales. Practically, this problem is often cope with by decomposition of “real” (no-scale) field into slow (large-scale) and fast (small-scale) components.

$$\phi = V + v, \quad \text{where} \quad Mv = 0$$

(M denotes mathematical expectation.) In this approach the slow component V governs the equation for the fast component v and the even-order moments of v contribute to the equation for V.

From the other hand, as we know from both Kolmogorov's theory and RG approach, there are no absolute scales in hydrodynamics, except for dissipative scale and external scale (the size of the system). So, at least at this middle – in the Kolmogorov range – the equations should be scale-covariant. The structure which reveals here looks like a fiber bundle over $\mathbb{R}^D$ with leaves labeled by scale. The fluctuations of different scales may be dependent or independent for various physical situations; but at least some similarity should be present.

To construct a basic system on this bundle let us follow the ideas of *multi-resolution analysis* [7] Let us construct a system of functional subspaces $\{V_i : V_i \subset \mathcal{H}\}$, Where $\mathcal{H}$ is a space of physical observables. Let the system $\{V_i\}$ be is such, that

1. $$\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots, \quad (2)$$
2. $\overline{\cup_{j=0}^{\infty}} V_j = L^2(R)$
3. $\cap_{j\in\mathbb{Z}} V_j = \emptyset$
4. The spaces V_j and V_{j-1} are ”similar”:

$$f(x) \in V_j \Leftrightarrow f(ax) \in V_{j+1}, \quad j \in \mathbb{Z},$$

where a is certain scale factor, often fixed as $a = 2$.

If the sequence $\{V_i\}$ is bounded from above, the maximal subspace is called the highest resolution space; let it be V_N. Then any function from V_{N-1} can be represented as a linear span of V_N basic vectors. Therefore, the basis ϕ_N of the highest resolution space provides a basis for a whole bundle.

It seems attractive to generalize MRA axioms to the case of continuous set of scales. Since the chain of subspaces described above implies sequential coarse graining of the finest resolution field, some details are being lost in course of this process. The lost details can be stored into the set of orthogonal complements

$$V_N = V_{N-1} \oplus W_{N-1}, \quad V_{N-1} = V_{N-2} \oplus W_{N-2}, \ldots \quad (3)$$

So, $\mathcal{H} = \overline{\sum_k W_k}, W_k \cap W_j = \emptyset$ if $j \neq k$, and the system $\{W_i\}$ can be considered instead of $\{V_i\}$. The former has the structure of σ-algebra and thus is suitable integration.

The fact, that velocities ϕ_l and $\phi_{l'}$ live on different leaves suggest that their Fourier decomposition should be taken separately at their leaves

$$\phi_l(x) = \int \exp(-\imath k^{(l)}x^{(l)})\tilde{\phi}_l(k^{(l)})d\mu_L^{(l)}(k^{(l)}), \quad (4)$$

or some other care should be taken about it in order not to mix fluctuations with same wave vectors but contributing to different scales. The choice of the left-invariant measure $d\mu_L^{(l)}(k^{(l)})$ is restricted by the fact, that velocity components measured at certain scale are mainly concentrated close to this scale. So the measure can be expressed as

$$d\mu_L^{(l)}(k) = dkW(|l^{-1} - ak|),$$

where $W(x)$ vanish at $x \to \pm\infty$, a is a constant.

The decomposition (4) turns to be a kind of Gabor transformation [4]. The measure can $d\mu$ can be explicitly scale-dependent, since the probability spaces $(\Omega_l, \mathcal{U}_l, P_l)$ are scale-dependent and their structure may be defined separately for each scale.

At this point we have arrived to the difference from standard wavelet approach, where the probability space is completely determined at finest resolution scale.

3 Wavelet realization

We start our construction of multi-scale description with simplistic one-dimensional case, which is however of practical importance since only one component of velocity field is often measured.

Any square-integrable function $f(t) \in L^2(R)$ can be represented as a decomposition with respect to the representations of affine group

$$t' = at + b \quad (5)$$

$$f(t) = C_\psi^{-1}\int \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)W_\psi(a,b)\frac{dadb}{a^2}, \quad (6)$$

which is just another form of the partition of unity with respect to representation of affine group acting on a Hilbert space $\mathcal{H}$ [5]

$$\hat{1} = C_\psi^{-1} \int_G U^*(g)|\psi\rangle d\mu_L(g)\langle\psi|U(g),$$

which holds if there exists such $\psi \in \mathcal{H}$, that

$$C_\psi = \frac{1}{\|\psi\|_2^2} \int_G |\langle\psi, |U(g)\psi\rangle|^2 d\mu_L(g) < \infty;$$

$d\mu_L(g)$ denotes left-invariant measure on G. The scalar products $W_\psi(g)f \equiv \langle f, U(g)\psi\rangle$ are known as wavelet coefficients, $\mathcal{H} = L^2(R)$ in our case.

For the case of affine transformation group (5) the normalization constant C_ψ can be easily evaluated in Fourier space

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\tilde{\psi}(k)|}{k} dk = 2\int_0^{\infty} \frac{|\tilde{\psi}(k)|}{k} dk \tag{7}$$

where $\psi(t) = \frac{1}{2\pi}\int \exp(\imath kt)\tilde{\psi}(k)dk$. For the affine group (5)

$$U(a,b)\psi(x) := \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right); \quad d\mu_L(a,b) = \frac{dadb}{a^2} \cdot \tag{8}$$

The corresponding wavelet coefficients are

$$W_\psi(a,b) = \int \overline{\frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)} f(t)dt. \tag{9}$$

For practical analytical calculations it is often more efficient to calculate wavelet coefficients in Fourier representation, since multiplication should be done then instead of convolution:

$$W_\psi(a,b)f = \frac{1}{2\pi}\int \sqrt{a}\exp(\imath kb)\overline{\tilde{\psi}(ak)}\tilde{f}(k)dk \tag{10}$$

and similar for reconstruction (6). The decomposition (9) and its inverse (6) are known as wavelet analysis. (See *e.g.* [1] for general review.) The scalar product (9) is readily seen to be the projection of the original "no-scale" function f to the subspace W_a of MRA system 3.

If f is a *random* function defined on a probability space $(\Omega, \mathcal{A}, P)$ the wavelet coefficients

$$W_\psi(a,b,\cdot) = \int \overline{\frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)} f(t,\cdot)dt \tag{11}$$

are also random; the stochastic integration is implied. As it is known from the theory of stochastic processes, a random function $\xi(t,\omega), t \in I\!R^1, \omega \in \Omega$ can be represented in a spectral form

$$\xi(t) = \int \phi(t,\lambda)\eta(d\lambda), \tag{12}$$

where $\phi(t,\lambda)$ is a square-integrable function, $\eta(d\lambda)$ is a stochastic measure

$$M\eta(d\lambda) = 0, \quad M|\eta(d\lambda)|^2 = F(d\lambda).$$

The particular form of the spectral representation (12) is Fourier representation

$$\xi(t) = \int_{-\infty}^{\infty} \exp(\imath\lambda t)\eta(d\lambda).$$

In a multi-scale case we can introduce a collection of random processes, each of them belonging to its own leaf of MRA, labeled by resolution parameter a

$$\xi_a(t) = \int \phi_a(t,\lambda)\eta_a(d\lambda).$$

The peculiarity of stochastic case is, that in contrast to the decomposition of a function with respect to given basic wavelet $\psi(t)$, the function $\phi_a(t,\lambda)$, which depends on both the properties of random process itself and filtering properties of measuring equipment, is not known exactly. Therefore, we have to construct a decomposition, which has a well defined limit to deterministic case and can be tackled without exact knowledge of the form of basic wavelet.

To do it we can factorize the scaling part of the left-invariant measure from "purely stochastic part"

$$\xi(t) = \int \phi_a(t,\lambda)\eta_a(d\lambda)\frac{da}{a},$$

or in the spectral form

$$\xi(t) = \frac{1}{2\pi C_\psi}\int e^{\imath\lambda t}\tilde{\psi}(\lambda a)\eta_a(d\lambda)\frac{da}{a}, \tag{13}$$

where $\eta_a(d\lambda)$ can be considered as generalized wavelet coefficients, the existence of which does not require the existence of "no-scale" prototype. The left invariant measure $d\mu(a) = da/(2\pi a)$ on the multiplicative group $x' = ax$ instead of (5), since translations are already incorporated into the exponent.

The representation (13) was constructed only to meet the non-stochastic limit and is not unique. For instance, we can redefine the spectral measure to incorporate both the properties of the signal and that of measuring apparatus

$$\xi(t) = \int e^{\imath\lambda t}\hat{\eta}_a(d\lambda)\frac{da}{a} \cdot$$

The specific energy-per-scale density can be defined from the decomposition of the total energy $E(\cdot) = \int \xi(t,\cdot)\bar{\xi}(t,\cdot)dt$ in terms of the wavelet images of the signal

$$E(\cdot) = C_\psi^{-1}\int |W_\psi(a,b,\cdot)|^2\frac{dadb}{a^2} \equiv \int E(a,\cdot)da.$$

Since $\xi(t,\cdot)$ is a random function its wavelet images $W_\psi(a,b,\cdot)$, the integrals $E(a,\cdot), E(\cdot)$ and their Fourier transforms are also random.

Taking the mathematical expectation of the both sides of the equality $E(\cdot) = \int E(a,\cdot)da$, where

$$E(a,\cdot) = \frac{1}{2\pi C_\psi}\int \frac{|\hat{W}_\psi(a,k,\cdot)|^2}{a}dk$$

we arrive at

$$ME(\cdot) = \frac{1}{2\pi C_\psi}\int \frac{da}{a} M \int |\hat{W}_\psi(a,k)|^2 dk \qquad (14)$$

or

$$\frac{\partial \mathcal{E}}{\partial \ln a} = \frac{1}{2\pi C_\psi} M \int |\hat{W}(a,k)|^2 dk \qquad (15)$$

where $\mathcal{E} \equiv ME(\cdot)$. The latter equation (15) has the form of the RG equation for the energy in quantum field theory. The components $\hat{W}(a,k)$, initially defined as a ψ-filtered initial signal $\xi(k,\cdot)$, $\hat{W}(a,k) = \overline{\tilde{\psi}(ak)}\tilde{\xi}(k)$, in our approach can be understood as the original noisy signal ξ perceived by filter ψ, *i.e.* as generalized wavelet coefficients, not necessary having "no-scale" prototypes, *cf.* equation (13).

4 Differences from standard wavelet approach

To clarify the differences from standard wavelet approach let us start with the MRA framework (2).

Let $V_j, j \in \mathbb{Z}$ be a nested family of subsets of $L^2(R)$ obeying (2).

The wavelet decomposition is performed by sequential projection of the finest resolution sample $f_N \in V_N$ of the function f by means of projection operator to projections on orthogonal complements $W_j = V_{j+1} \setminus V_j$, since $V_{j+1} = V_j \oplus W_j$. This decomposition has the form

$$\begin{aligned} f_{j+1} &= g_j + f_j = g_j + g_{j-1} + f_{j-1} \\ &= g_j + \ldots + g_{j-m} + f_{j-m}, \end{aligned} \qquad (16)$$

where

$$f_j := P_j f, \quad g_j = (P_{j+1} - P_j)f := Q_j f.$$

If the function f describes the measured experimental signal, then there is a finest resolution sample $f_N, (N < \infty)$. f_{N-1} can be considered as first blurring of f_N, and g_{N-1} accounts for the details neglected in this blurring, *cf.* (16). Schematically, the process is presented at the diagram bellow.

$$\begin{array}{ccccccc} f_N & \to & f_{N-1} & \to & f_{N-2} & \to & \ldots \\ & \searrow & & \searrow & & \searrow & \\ & & g_{N-1} & & g_{N-2} & & \ldots \end{array} \qquad (17)$$

Since in each blurring $f_k \to f_{k-1}$ some information is lost, for sufficiently large m the f_{k-m} projections can be regarded as insignificant constants – the whole information is therefore stored in the sequence of projections onto W spaces

$$\{g_{n-1}, g_{n-2}, \ldots g_{n-m}\}. \qquad (18)$$

The function f_{n-k} may be considered as the measurement of the process f at kth resolution level, where $k = 0$ stands for finest possible resolution. Since $f_{k-1} = Pf_k$, the process has truly renormalization group nature, *if* P is a constant operator. Physically, the process of measuring always implies certain averaging (quantum aspects are not considered here), and the measuring at different resolution levels may obey different laws

$$f_{k-1} = P_k f_k, \quad f_{m-1} = P_m f_m, P_k \neq P_m.$$

The finest resolution level $f_{n=\infty}$ simply may not exist, *e.g.* it is meaningless to to speak about velocity of fluid flow at 10^{-8} cm scales. In this case the sequence (18) has its own meaning and is the unique representative of the measured process.

$$f_N := g_{N-1} + \ldots + g_{N-M} + const.,$$

or in continuous approximation:

$$f_A(x) = C_\psi^{-1} \int_A^\infty W_\psi(a,b) a^{-1/2} \psi\left(\frac{x-b}{a}\right)\frac{dadb}{a^2} \cdot \qquad (19)$$

Thus, instead of a single function $f \in L^2(R)$ we need a family of functions $\{f_A\}_A$, indexed by resolution parameter A.

To summarize it briefly, in wavelet approach the decomposition of a certain function $f \in L^2(R)$ is performed, in our approach the indexed family (19) is used without any requirements on existence of f_∞.

5 High frequency cutoff

Being utmost scale-invariant at moderate scales, the behavior of turbulent velocity field changes when approaching the smallest and largest scales, between which the hydrodynamical description is valid. The former is the Kolmogorov dissipative scale (η), the latter is the size of the system. The size of the system can often be set to infinity with no harm to physics; whilst the dissipative scale is of physical importance, since the energy dissipation rate $\bar{\epsilon}$ is very constant which determines the turbulence behavior in inertial range.

That is why in RG, as well as in spectral calculations, the cutoff dependent velocity field is often considered

$$v_F^<(x) = \frac{1}{(2\pi)^d}\int_{|k|<F} \exp(\imath kx)\tilde{v}(k)dk. \qquad (20)$$

The cumulative energy of all harmonics with wave vectors less or equal to the cut-off value F is one of the main spectral characteristics of developed turbulence

$$\begin{aligned} \mathcal{E}(F) &= \frac{1}{2} M \int \overline{v_F^<(x)} v_F^<(x) d^d x \\ &= \frac{1}{2} M \int_{|k|<F} \overline{\tilde{v}(k)}\tilde{v}(k)\frac{d^d k}{(2\pi)^d} \cdot \end{aligned} \qquad (21)$$

Similarly, we can consider the cumulative energy of all velocity fluctuations with typical size greater or equal to a given A. For simplicity let us consider a one-component velocity field considered as a function of time

$$\begin{aligned} E(A) &= \frac{1}{2} M \int_{|a|\ge A} \overline{v(t)} v(t) dt \\ &= \frac{1}{C_\psi} M \int_{a=A}^{\infty} |W_\psi(a,b)v|^2 \frac{dadb}{a^2} \\ &= C_\psi^{-1} \int_A^\infty \frac{|\tilde\psi(y)|^2}{y} dy \cdot M \int |\tilde v(k)|^2 \frac{dk}{2\pi} \end{aligned} \quad (22)$$

where

$$\lim_{A\to 0} 2 \int_{a=A}^{\infty} \frac{|\tilde\psi(y)|^2}{y} dy = C_\psi \quad \text{and} \quad E = \frac{1}{2} M \int |\tilde v(k)|^2 \frac{dk}{2\pi}$$

is the total energy of all velocity fluctuations.

For non-vanishing A

$$E(A) = F(A)E,$$

where

$$F(A) = \frac{\int_A^\infty \frac{|\tilde\psi(y)|^2}{y} dy}{\int_0^\infty \frac{|\tilde\psi(y)|^2}{y} dy} \cdot \quad (23)$$

For definiteness, let us calculate the filtering function $F(A)$ for a particular family of vanishing momenta wavelets (often used for studying of hydrodynamical velocity field [8,9]).

$$\begin{aligned} \psi_n(x) &= (-1)^n \frac{d^n}{dx^n} \exp(-x^2/2), \\ \tilde\psi_n(k) &= \sqrt{2\pi}(-\imath k)^n \exp(-k^2/2). \end{aligned} \quad (24)$$

The normalization constant for this family is $C_n = 2\pi \int_{-\infty}^{\infty} k^{2n-1} e^{-k^2} dk = 2\pi\Gamma(n)$ and so

$$F_n(A) = \frac{\int_{A^2}^\infty y^{n-1} e^{-y} dy}{\Gamma(n)} \cdot \quad (25)$$

The derivative of cumulative energy with respect to logarithmic measure da/a is

$$\frac{\partial E}{\partial \ln A} = E \frac{\partial F_n(E)}{\partial \ln A} = -\frac{\partial A^2}{\partial \ln A} f_n(A^2)$$

where $f_n(x) = x^{n-1} e^{-x}/\Gamma(n)$. So we arrive at RG like equation

$$\frac{\partial E}{\partial \ln A} = -\frac{2A^{2n} \exp(-A^2)}{\Gamma(n)} E. \quad (26)$$

For sufficiently small A the exponential term is close to unity, and thus the behavior is approximately proportional to A^2. This is a power law behavior of the energy, similar to that calculated by field theory methods in hydrodynamic turbulence theory [3].

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References

1. I. Daubechies, Comm. Pure Appl. Math. **41**, 909 (1988).
2. B. Dubrulle, F. Graner, J. Phys. II France **6**, 797 (1996).
3. D. Forster, D.R. Nelson, M.J. Stephen, Phys. Rev. A **16**, 732 (1977).
4. D. Gabor, Proc. IEE **93**, 429 (1946).
5. A. Grossmann, J. Morlet, T. Paul, J. Math. Phys. **26**, 2473 (1985).
6. A.N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 9 (1941); Reprinted in Proc. R. Soc. Lond. A **434**, 9 (1991).
7. S. Mallat, A theory for multiresolution signal decomposition: wavelet transform, Preprint GRASP Lab. Dept. of Computer an Information Science, University of Pennsilvania (1986).
8. J.F. Muzy, E. Bacry, A. Arneodo, Phys. Rev. Lett. **67**, 3515 (1991).
9. J.F. Muzy, E. Bacry, A. Arneodo, Phys. Rev. E **47**, 875 (1993).